ALLIGATION, FORERUNNER OF LINEAR PROGRAMMING

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Mathematics is the queen of science, and arithmetic is the queen of mathematics—Gauss.

IF HERR DOKTOR PROFESSOR GAUSS could read the present-day scientific journals, he would find much evidence that mathematics is, indeed, revered as the queen of science. But he might wonder if we now recognize arithmetic as the queen of mathematics. The modern science of economics, for example, makes much use of matrix inversion, of difference equations, and of other forms of "higher" mathematics. But arithmetic gets little attention.[2]

A case in point is linear programming. The methods developed by Dantzig[3] and others[4] are elegant, but they are hard for the ordinary economist to understand. Several authors[5] have tried to help him by publishing simplified explanations of these methods. But perhaps there is another alternative. Perhaps the economist will find that simple arithmetic methods will enable him to solve typical linear programming problems. He doesn't necessarily have to be able to manipulate matrices and vectors if he has a good grounding in arithmetic.

Unfortunately, though, we are not getting as good training in arithmetic as our parents and grandparents got. Among the many branches of arithmetic formerly taught was "alligation," which is a simple form of linear programming. Regrettably, the subject of alligation seems to have been dropped from the arithmetic books about the turn of the twentieth century.[6]

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[2] Such books as Allen, R. G. D., *Mathematics for Economists,* Macmillan and Co., London, 1947, seem to assume that the economist already knows all about arithmetic.

[3] Dantzig, G. B., "Maximization of a Linear Form Where Variables are Subject to a System of Linear Inequalities." Dittoed. *USAF Comptroller,* Nov. 1949.

[4] Koopmans, T. C., ed., *Activity Analysis of Production and Allocation,* John Wiley & Sons, New York, 1951.

[5] Charnes, A., Cooper, W. W., and Henderson, A., *An Introduction to Linear Programming,* John Wiley & Sons, New York, 1953.

Heady, Earl, "Simplified Presentation and Logical Aspects of Linear Programming." *Journal of Farm Economics.* Dec. 1954.

Waugh, F., and Burrows, G., "A Short-cut to Linear Programming." *Econometrica,* Jan. 1955.

[6] My mother-in-law, Mrs. L. F. Wilcox, could beat most modern mathematicians

In this paper, I shall first describe the principles of alligation, quoting from Pike and from Ray.[7] Then I shall show how these principles can be extended slightly and used to analyze a fairly complicated problem of linear programming, dealing with feeds for broilers (young chickens).

Definitions

In the words of Mr. Pike:

Alligation is the method of mixing two or more simples of different qualities, so that the composition may be of a mean or middle quality.

The "simples" are the original ingredients to be mixed. In our broiler-feed problem, the simples are such ingredients as soybean meal, corn, and calcium carbonate. The purpose of alligation is to find a mixture of several ingredients that will meet stated specifications—i.e., a mixture that will have a stated mean quality. This is essentially the problem of linear programming. Alligation will discover one or more "feasible solutions," if they exist. It alone will not necessarily discover the "minimum-cost feasible solution."

Mr. Pike goes on to distinguish two kinds of alligation. In his words,

Alligation Medial is, when the quantities and prices of several things are given, to find the mean price of the mixture compounded from those things.

Alligation Alternate is the method of finding what quantity of each of the ingredients, whose rates are given, will compose a mixture of a given rate: so that it is the reverse of Alligation Medial, and may be proved by it.

Pike's *Arithmetick* gives long and precise rules for computing both kinds of alligation. Rather than quote these rules in detail, I propose to illustrate them by a few numerical examples taken from the broiler-feed study.

Alligation of Pairs of Ingredients

The general principle of alligation alternate can be illustrated best in problems involving only two ingredients. Pike does not analyze such a simple problem. I shall give two examples from Ray, and also an example given to me by Ronald Mighell.

The first example from Ray is:

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at arithmetic, which she learned in the grade schools in upper New York State. She has fondly preserved a few old texts, including Pike's *Arithmetick* and Ray's *New Higher Arithmetic*.

Pike, Nicolas, *New and Complete System of Arithmetick*, 7th ed., Revised, corrected, and improved, and more particularly adapted to the Federal Currency (Boston: Thomas & Andrews, 1809).

Ray, Joseph, *New Higher Arithmetic* (Cincinnati and New York: Van Antwerp, Bragg and Co., 1880).

[7] *Op. cit.*

PROBLEM.—*What relative quantities of sugar, at 9 ct. a lb. and 5 ct. a lb., must be used for a compound, at 6 ct. a lb.?*
SOLUTION.—

Operation

$$6\left|\begin{array}{c}5\\ \overline{9}\end{array}\right) \begin{array}{ll} 3 \text{ lb. at 5 ct.} = & 15 \text{ ct.} \\ \underline{1 \text{ lb. at 9 ct.}} = & \underline{\ 9 \text{ ct.}} \\ 4 \text{ lb. worth} & 24 \text{ ct.} \end{array}$$

$$\text{which is } \frac{24}{4} = 6 \text{ ct. a lb.}$$

The "operation" is simple. Take the absolute differences between the given prices and the mean of 6 cents. Thus $6 - 5 = 1$, and $9 - 6 = 3$. Reverse the order of these numbers. That is, mix 1 pound at 9 cents with 3 pounds at 5 cents. To the right of the little table, Mr. Ray proves his answer by the use of alligation medial.

Why does this arithmetic work? Mr. Ray says,

If you put 1 lb. at 9 ct. in the mixture to be sold at 6ct., you lose 3 ct.; if you put 1 lb. at 5 ct. in the mixture to be sold at 6 ct., you gain 1 ct.; 3 such lb. gain 3 ct.; the gain and loss would then be equal if 3 lb. at 5 ct. are mixed with 1 lb. at 9ct.

Mr. Pike made a similar explanation:

By connecting the less rate with the greater, and placing the difference between them and the mean rate alternately, or one after the other in turn, the quantities are such, that there is precisely as much gained by the one quantity as is lost by the other, and therefore the gain and loss upon the whole are equal, and are exactly the proposed rate.

Perhaps most readers are so far removed from arithmetic that they would find an algebraic explanation more convincing. Suppose we were to mix two ingredients–the first costing a cents a pound, and the second costing b cents a pound, in such proportions that the mixture will cost c cents a pound. Assube that $a<c<b$. Mix (b–c) pounds of the first ingredient with (c–a) pounds of the second. The total cost will be $a\,(b-c) + b\,(c-a) = c\,(b-a)$ cents. The total number of pounds in the mixture is $(b-c) + (c-a) = b-a$. So the average cost of the mixture is

$$\frac{c(b - a)}{b - a} = c,$$

as it should be.

The first example concerned the average price of a mixture. The same arithmetic method can be used to mix a pair of ingredients to get a stated average quality. Another example from Ray:

PROBLEM.—*What of silver $\frac{3}{4}$ pure, and $\frac{9}{10}$ pure, will make a mixture $\frac{7}{8}$ pure?*

First restate the fractions as $\frac{30}{40}$, $\frac{36}{40}$, and $\frac{35}{40}$.

					OPERATION
SOLUTION	$\frac{35}{40}$	$\frac{30}{40}$	$\frac{1}{40}$	1 lb.,	$\frac{30}{40}$ pure $=$ $\frac{30}{40}$ lb. pure silver
		$\frac{36}{40}$	$\frac{5}{40}$	5 lbs.,	$\frac{36}{40}$ pure $=$ $\frac{180}{40}$ lb. pure silver
			Total	6 lbs.	Total $\frac{210}{40}$ lb. pure silver.

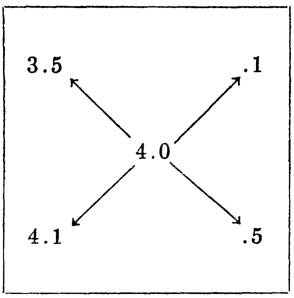
Thus, one pound of the mixture contains $\frac{210}{6 \times 40} = \frac{210}{240} = \frac{7}{8}$ lb. pure silver, as it should.

Note that the second column indicates a mixture of $\frac{1}{40}$ lb. and $\frac{5}{40}$ lb. of the two lots of silver. But in this (and in most) examples, we are interested only in proportions. So I have indicated a mixture of 1 lb. and 5 lbs.

Before leaving mixtures of two ingredients, I shall quote a recent note from Mighell:[8]

"In a Freshman course in Dairy Manufactures we were taught to use the following device for standardizing milk or cream to a given percentage of butterfat:

Suppose we had some 3.5 per cent, and some 4.1 per cent milk. How could they be mixed to make 4.0 per cent? We drew a square and placed the percentages as indicated. The diagonal differences then indicated the horizontal proportions of each test of milk. . . . This particular scheme may have some advantage for remembering a simple procedure."



This "particular scheme" is, of course, a different way to record the results of alligation alternate. The name has been lost, but the practice lingers on.

Alligation of Three or More Ingredients

Take the following example from Pike:

A merchant has Canary wine, at 3s. per gallon, Sherry at 2s. 1d., and Claret at 1s. 5d. per gallon: How much of each sort must he take to sell it at 2s. 4d. per gallon?

[8] Note from Dr. Ronald L. Mighell of the Agricultural Research Service, USDA to the author, October 7, 1957.